MATH 2050C Lecture 21 (Mar 31)

* No lectures / tutorials next week (Reading week). *

[Problem Set 11 posted, due on Apr 15.]

Three important theorems about continuous $f:[a,b]\to\mathbb{R}$

- Boundedness Thm
- Extreme Value Theorem.
- Intermediate Value Theorem

Boundedness $\Rightarrow$ $\exists\, M>0$ s.t. $|f(x)|\leq M \quad \forall x\in[a,b]$

EVT $\Rightarrow$ max f & min f are "achieved".

Intermediate Value Theorem

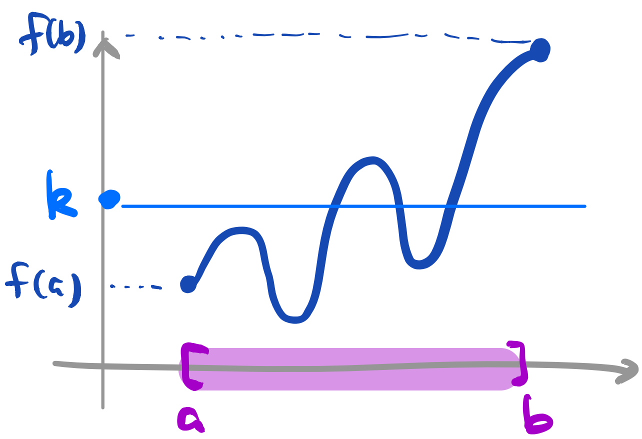
Let $f:[a,b]\to\mathbb{R}$ be a cts function s.t.

$$f(a)<f(b)$$

THEN: $\forall\, k\in(f(a),f(b))$, $\exists\, c\in[a,b]$ s.t.

$$f(c)=k$$

Proof: Picture:

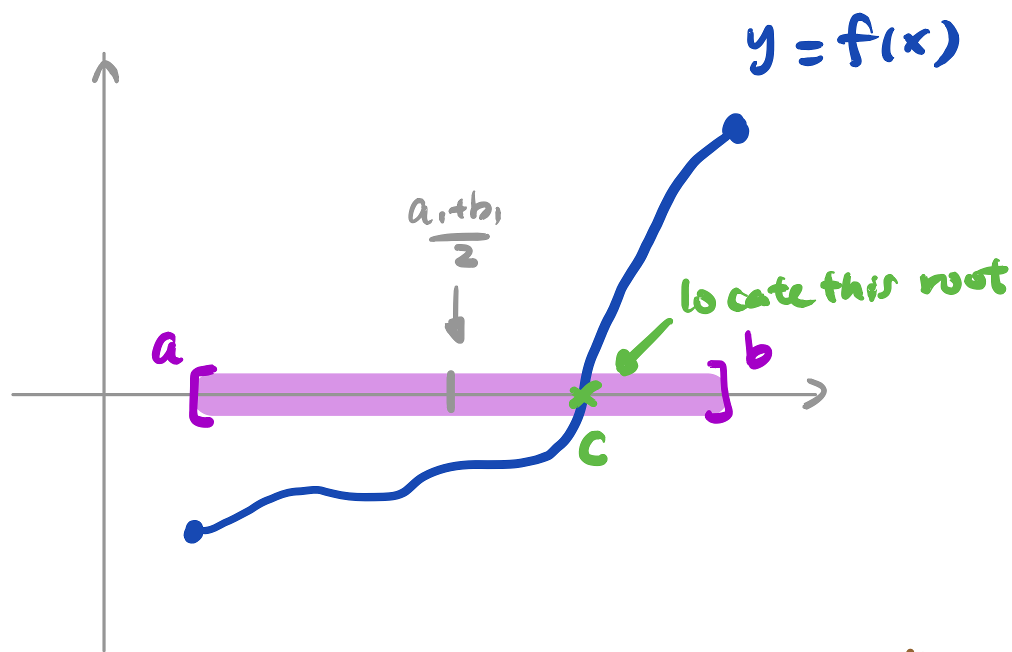


Fix k. **WLOG**: Assume $\boxed{k=0}$, $f(a) < 0 < f(b)$.

Reason: Consider $g(x) := f(x) - k$ cts on $[a,b]$

and $g(c) = 0 \iff f(c) = k$

GOAL: Locate a "root" to $f(x) = 0$.



We will use "Method of Bisection"; and we will show that it works using Nested Interval Property. We proceed inductively as follows:

Define: $I_1 := [a, b] := [a_1, b_1]$

Consider the mid pt. $\frac{a_1+b_1}{2}$ of I_1

Case 1: $f\left(\frac{a_1+b_1}{2}\right) < 0 \Rightarrow I_2 := \left[\frac{a_1+b_1}{2}, b_1\right] := [a_2, b_2]$

Case 2: $f\left(\frac{a_1+b_1}{2}\right) > 0 \Rightarrow I_2 := \left[a_1, \frac{a_1+b_1}{2}\right] := [a_2, b_2]$

Case 3: $f\left(\frac{a_1+b_1}{2}\right) = 0 \Rightarrow$ DONE! $c = \frac{a_1+b_1}{2}$.

Repeat the process for I_2 etc.

Throughout this iterative process.

either: you locate a root at a finite step.

$\Rightarrow$ DONE!

or: this goes on forever

$\Rightarrow \exists$ a nested seq $I_n := [a_n, b_n]$ of closed and bad intervals s.t.

bisection

- $\text{Length}(I_{n+1}) = \frac{1}{2}\text{Length}(I_n)$

(*) — • $f(a_n) < 0 < f(b_n) \quad \forall n \in \mathbb{N}$

by construction

N.I.P. $\Rightarrow \bigcap_{n=1}^{\infty} I_n = \{c\}$, i.e. $\lim(a_n) = c = \lim(b_n)$

Claim: $f(c) = 0$

Pf: f cts, take $n \to \infty$ in (*).

$$f(c) = \lim_{n\to\infty} f(a_n) \le 0 \le \lim_{n\to\infty} f(b_n) = f(c)$$

$\square$

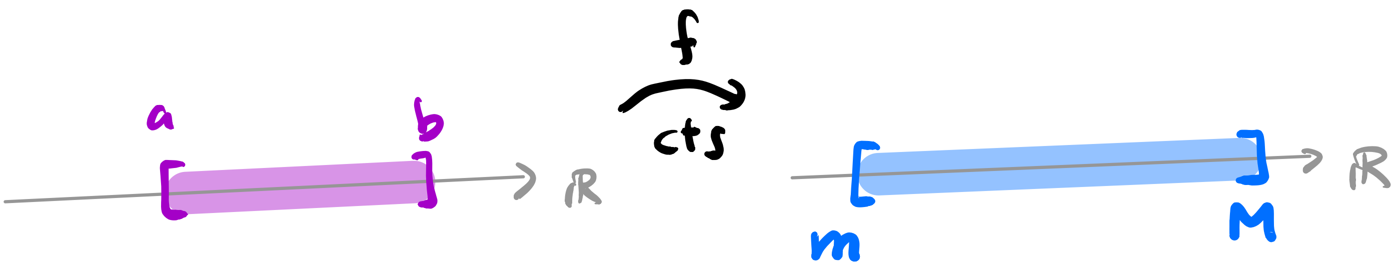
So, we have established:

Three important theorems about continuous $f:[a,b]\to\mathbb{R}$
$\left\{\begin{array}{l}\text{Boundedness Thm} \\ \text{Extreme Value Theorem.} \\ \text{Intermediate Value Theorem}\end{array}\right.$

Cor: If $f:[a,b]\to\mathbb{R}$ is cts, then

$$f([a,b]) := \{f(x) : x\in[a,b]\} = [m,M].$$

here $m = \inf_{x\in[a,b]} f(x)$ and $M = \sup_{x\in[a,b]} f(x)$



i.e. cts functions takes closed & bdd intervals to closed & bdd intervals.

"Topologically" (MATH 3070)

cts f preserves "compactness" &

"connectedness"